

Advanced Probability

Notes based on Book: *Probability: Theory and Examples* by Rick Durrett

Last updated: **September 13, 2025**

Contents

| | |
|---|---|
| Chapter 1: Measure Theory | 3 |
| 1.1 Probability Spaces | 3 |
| 1.2 Distributions | 3 |
| 1.3 Random Variables | 3 |
| 1.4 Integration | 3 |
| 1.5 Properties of the Integral | 3 |
| 1.6 Expected Value | 3 |
| 1.6.1 Inequalities | 3 |
| 1.6.2 Integration to the Limit | 3 |
| 1.6.3 Computing Expected Values | 3 |
| 1.7 Product Measures, Fubini's Theorem | 3 |
| Chapter 2: Laws of Large Numbers | 3 |
| 2.1 Independence | 3 |
| 2.1.1 Sufficient Conditions for Independence | 3 |
| 2.1.2 Independence, Distribution, and Expectation | 3 |
| 2.1.3 Sums of Independent Random Variables | 3 |
| 2.1.4 Constructing Independent Random Variables | 3 |
| 2.2 Weak Laws of Large Numbers | 3 |
| 2.2.1 L^2 Weak Laws | 3 |
| 2.2.2 Triangular Arrays | 3 |
| 2.2.3 Truncation | 3 |
| 2.3 Borel-Cantelli Lemmas | 3 |
| 2.4 Strong Law of Large Numbers | 3 |
| 2.5 Convergence of Random Series* | 3 |
| 2.5.1 Rates of Convergence | 3 |
| 2.5.2 Infinite Mean | 3 |
| 2.6 Renewal Theory* | 3 |
| 2.7 Large Deviations* | 3 |
| Chapter 3: Central Limit Theorems | 3 |
| 3.1 The De Moivre-Laplace Theorem | 3 |
| 3.2 Weak Convergence | 3 |
| 3.3 Characteristic Functions | 3 |
| 3.3.1 Definition, Inversion Formula | 3 |
| 3.3.2 Weak Convergence | 3 |
| 3.3.3 Moments and Derivatives | 3 |
| 3.3.4 Polya's Criterion* | 3 |
| 3.3.5 The Moment Problem* | 4 |
| 3.4 Central Limit Theorems | 4 |
| 3.4.1 i.i.d. Sequences | 4 |
| 3.4.2 Triangular Arrays | 4 |
| 3.4.3 Prime Divisors (Erdős-Kac)* | 4 |
| 3.4.4 Rates of Convergence (Berry-Esseen)* | 4 |
| 3.5 Local Limit Theorems* | 4 |
| 3.6 Poisson Convergence | 4 |

| | | |
|------------|---|---|
| 3.6.1 | The Basic Limit Theorem | 4 |
| 3.6.2 | Two Examples with Dependence | 4 |
| 3.7 | Poisson Processes | 4 |
| 3.7.1 | Compound Poisson Processes | 4 |
| 3.7.2 | Thinning | 4 |
| 3.7.3 | Conditioning | 4 |
| 3.8 | Stable Laws* | 4 |
| 3.9 | Infinitely Divisible Distributions* | 4 |
| 3.10 | Limit Theorems in \mathbb{R}^d | 4 |
| Chapter 4: | Martingales | 4 |
| 4.1 | Conditional Expectation | 4 |
| 4.1.1 | Examples | 4 |
| 4.1.2 | Properties | 4 |
| 4.1.3 | Regular Conditional Probabilities* | 4 |
| 4.2 | Martingales, Almost Sure Convergence | 4 |
| 4.3 | Examples | 4 |
| 4.3.1 | Bounded Increments | 4 |
| 4.3.2 | Polya's Urn Scheme | 4 |
| 4.3.3 | Radon-Nikodym Derivatives | 4 |
| 4.3.4 | Branching Processes | 4 |
| 4.4 | Doob's Inequality, Convergence in L^p , $p > 1$ | 4 |
| 4.5 | Square Integrable Martingales* | 4 |
| 4.6 | Uniform Integrability, Convergence in L^1 | 4 |
| 4.7 | Backwards Martingales | 4 |
| 4.8 | Optional Stopping Theorems | 4 |
| 4.8.1 | Applications to Random Walks | 4 |
| 4.9 | Combinatorics of Simple Random Walk* | 4 |
| Chapter 5: | Markov Chains | 4 |
| 5.1 | Examples | 5 |
| 5.2 | Construction, Markov Properties | 5 |
| 5.3 | Recurrence and Transience | 5 |
| 5.4 | Stationary Measures | 5 |
| 5.5 | Asymptotic Behavior | 5 |
| 5.6 | Periodicity, Tail σ -Field* | 5 |
| 5.7 | General State Space* | 5 |
| 5.7.1 | Recurrence and Transience | 5 |
| 5.7.2 | Stationary Measures | 5 |
| 5.7.3 | Convergence Theorem | 5 |
| 5.7.4 | GI/G/1 Queue | 5 |

Chapter 1: Measure Theory**1.1 Probability Spaces****1.2 Distributions****1.3 Random Variables****1.4 Integration****1.5 Properties of the Integral****1.6 Expected Value****1.6.1 Inequalities****1.6.2 Integration to the Limit****1.6.3 Computing Expected Values****1.7 Product Measures, Fubini's Theorem****Chapter 2: Laws of Large Numbers****2.1 Independence****2.1.1 Sufficient Conditions for Independence****2.1.2 Independence, Distribution, and Expectation****2.1.3 Sums of Independent Random Variables****2.1.4 Constructing Independent Random Variables****2.2 Weak Laws of Large Numbers****2.2.1 L^2 Weak Laws****2.2.2 Triangular Arrays****2.2.3 Truncation****2.3 Borel-Cantelli Lemmas****2.4 Strong Law of Large Numbers****2.5 Convergence of Random Series*****2.5.1 Rates of Convergence****2.5.2 Infinite Mean****2.6 Renewal Theory*****2.7 Large Deviations*****Chapter 3: Central Limit Theorems****3.1 The De Moivre-Laplace Theorem****3.2 Weak Convergence****3.3 Characteristic Functions****3.3.1 Definition, Inversion Formula****3.3.2 Weak Convergence****3.3.3 Moments and Derivatives****3.3.4 Polya's Criterion***

3.3.5 The Moment Problem***3.4 Central Limit Theorems****3.4.1 i.i.d. Sequences****3.4.2 Triangular Arrays****3.4.3 Prime Divisors (Erdős-Kac)*****3.4.4 Rates of Convergence (Berry-Esseen)*****3.5 Local Limit Theorems*****3.6 Poisson Convergence****3.6.1 The Basic Limit Theorem****3.6.2 Two Examples with Dependence****3.7 Poisson Processes****3.7.1 Compound Poisson Processes****3.7.2 Thinning****3.7.3 Conditioning****3.8 Stable Laws*****3.9 Infinitely Divisible Distributions*****3.10 Limit Theorems in \mathbb{R}^d** **Chapter 4: Martingales****4.1 Conditional Expectation****4.1.1 Examples****4.1.2 Properties****4.1.3 Regular Conditional Probabilities*****4.2 Martingales, Almost Sure Convergence****4.3 Examples****4.3.1 Bounded Increments****4.3.2 Polya's Urn Scheme****4.3.3 Radon-Nikodym Derivatives****4.3.4 Branching Processes****4.4 Doob's Inequality, Convergence in L^p , $p > 1$** **4.5 Square Integrable Martingales*****4.6 Uniform Integrability, Convergence in L^1** **4.7 Backwards Martingales****4.8 Optional Stopping Theorems****4.8.1 Applications to Random Walks****4.9 Combinatorics of Simple Random Walk*****Chapter 5: Markov Chains**

5.1 Examples**5.2 Construction, Markov Properties****5.3 Recurrence and Transience****5.4 Stationary Measures****5.5 Asymptotic Behavior****5.6 Periodicity, Tail σ -Field*****5.7 General State Space*****5.7.1 Recurrence and Transience****5.7.2 Stationary Measures****5.7.3 Convergence Theorem****5.7.4 GI/G/1 Queue**